Int. J. Solids Structures. 1975, Vol. 11, pp. 999–1010. Pergamon Press. Printed in Gt. Britain.

FINITE EXPANSION OF A HOLE IN AN ELASTIC-PLASTIC THIN PLATE OF FINITE THICKNESS

HIROYUKI ABÉ
Department of Mechanical Engineering, Tohoku University, Sendai, Japan

(*Received* 17 *October* 1974; *revised* 20 *January* 1975)

Abstract—A finite deformation theory for axially symmetric thin elastic-plastic plates is obtained by a consistent approximation from the corresponding three-dimensional theory. This theory can be applied to plates of finite thickness, and is different from the usual plane stress theory. The problem of the expansion of a circular hole in an infinite plate is investigated and the effect of the transversely variable stress components is studied.

1. INTRODUCTION

The problem of the enlargement of a circular hole in an infinite plate of uniform initial thickness has been studied by Taylor[1], Hill[2], Prager[3] and Hodge and Sankaranarayanan[4]. Nordgren and Naghdi studied the problem of annular plates[5–6]. Similar problems for plates of non-uniform thickness have been considered by Alexander and Ford[7], Rogers[8], Nemat-Nasser[9], Chern[10] and Chern and Nemat-Nasser[11–13]. Mansfield solved the problem of a variable thickness sheet subjected to the radial stress at infinity[14].

The studies mentioned above are based on the plane stress theory which is formulated under the assumption that the transverse stresses can be ignored throughout the plate. This theory is adequate for a very thin plate. However, the theory for plane stress contains almost no mention of the ranges of application to actual plates of finite thickness. The transverse shear stress components are expected to appear in finite, especially variable thickness plates[15–17].

Any two-dimensional theory (one-dimensional theory for axially symmetric problems) is necessarily an approximation of the exact three-dimensional theory. The study of making consistent formulations of the two-dimensional theory and of finding its ranges of application, is important in practical problems. Although it is the best to use the three-dimensional solutions for finding the validity of the corresponding two-dimensional theory, the three-dimensional equations are often difficult to solve. The author has developed systematically a two-dimensional theory, starting from the three-dimensional equations, for the finite in-plane deformation of thin elastic-plastic plates of finite thickness in which the transverse shear deformability is taken into account[17]. It is a consistent extension of the usual plane stress theory.

We introduce parameters λ and λ_h, where λ denotes an equivalent thickness depending on the wave length of the deformation pattern on the middle plane, and λ_h is a function which characterizes the variation of the initial thickness. Further, we let ϵ_0 be the magnitude of strain. The parameter λ becomes larger as the values of the stress (or strain) gradient in the longitudinal direction and/or the ratio h_0/a increases; here h_0 is the initial thickness, and a is the smallest characteristic length of the middle plane. For plates with a large variation of the thickness h_0 and/or a large value of the ratio h_0/a, λ_h takes on a large value. The two-dimensional theory is found to be valid when $O(\epsilon_0, \lambda^2, \lambda_h^2) \leqq \lambda_B^2$, where λ_B is a certain universal constant[17]. If we consider the results given by Alexander and Ford[7] as an example, the stress concentrates rapidly near the rim of the hole so that h_0/a has to be sufficiently small in order to satisfy the above condition. The transverse shear stress may become important for problems within the usual ranges of the two-dimensional theory; it may even become of the same order to the in-plane stresses for plates with large λ_h.

In this paper, we consider the axisymmetric expansion of a circular hole in an infinite plate of uniform, or non-uniform, initial thickness using the two-dimensional finite deformation theory derived by the author[17]. The development is made in a Lagrangian frame of reference. The material of the plate is assumed to obey the von Mises yield criterion and the modified Prandtl–Reuss incremental relationship adopted by Hibbitt *et al.*[18]. Further, an isotropic hardening of the Ramberg–Osgood type is assumed here. First, we summarize the basic equations for axisymmetric problems of plates. A second order non-linear differential equation is derived in terms of the increment of the radial displacement component. Then we obtain iteratively solutions of the finite difference approximation of this equation.

2. BASIC EQUATIONS FOR AXIALLY SYMMETRICAL PROBLEMS

Although the greater part of this section depends largely on [17], it will be stated here for further use.

2.1 *Geometrical preliminaries*

Let (r, θ, z) denote a convected cylindrical coordinates and the plane $z = 0$ be taken as the undeformed middle plane of the plate, which we shall call the plane S_0. The undeformed plate is bounded by the surfaces $z = \pm h_0/2$ where h_0 may be a function of r.

We consider a plate to undergo a deformation symmetrical about the plane S_0. The square of the length ds_0 of a line element on the plane S_0 is given by $(ds_0)^2 = (dr)^2 + (rd\theta)^2$. In the deformed state the square of the length ds of the same material line element changes to

$$(ds)^2 = A_r(dr)^2 + A_\theta(rd\theta)^2, \tag{2.1}$$

where

$$A_r = \left(1+\frac{du}{dr}\right)^2, \qquad A_\theta = \left(1+\frac{u}{r}\right)^2. \tag{2.2}$$

The strain components in the Lagrangian coordinates are therefore expressed by

$$\epsilon_r = \frac{du}{dr}+\frac{1}{2}\left(\frac{du}{dr}\right)^2, \qquad \epsilon_\theta = \frac{u}{r}+\frac{1}{2}\left(\frac{u}{r}\right)^2, \qquad \gamma_{r\theta} = 0. \tag{2.3}$$

The initial volume element $d\tau_0$ changes to $d\tau = \sqrt{(G)}\, d\tau_0$ after deformation where

$$G = \left(\frac{d\tau}{d\tau_0}\right)^2 = (1+2\hat{r}_z)A_rA_\theta. \tag{2.4}$$

The quantity $\hat{r}_z$ is a coefficient relating to the transverse strains.

2.2 *Simplification from the three-dimensional theory*

Let s_r and s_θ/r^2 denote the Kirchhoff stress tensors at a point on the surface S_0. Further let L and L_h denote the wave lengths of the deformation pattern and of the variation of the undeformed thickness h_0, respectively, in the radial direction of the plate. We put

$$\begin{gathered}\epsilon_0 = \max_{S_0} \sqrt{(\epsilon_r^2+\epsilon_\theta^2)}, \qquad s_0 = \max_{S_0}\sqrt{(s_r^2+s_\theta^2)},\\ \delta_0 = \max_{S_0} |r_z|, \qquad \lambda = \max_{S_0}\frac{h_0}{L}, \qquad \lambda_h = \max_{S_0}\frac{h_0}{L_h},\end{gathered} \tag{2.5}†$$

where r_z is the value of $\hat{r}_z$ at $z = 0$. We also define the following symbols:

$$\begin{gathered}\epsilon_k = \max_{S_0} h_0{}^k\sqrt{[(\overset{(k)}{\epsilon}_r)^2+(\overset{(k)}{\epsilon}_\theta)^2]}, \qquad \gamma_k = \max_{S_0} h_0{}^k|\overset{(k)}{\gamma}_{rz}|, \qquad \delta_k = \max_{S_0} h_0{}^k|\overset{(k)}{\epsilon}_z|,\\ s_k = \max_{S_0} h_0{}^k\sqrt{[(\overset{(k)}{s}_r)^2+(\overset{(k)}{s}_\theta)^2]}, \qquad t_k = \max_{S_0} h_0{}^k|\overset{(k)}{s}_{rz}|,\\ u_k = \max_{S_0} h_0{}^k|\overset{(k)}{s}_{zz}|, \quad (k \text{ not summed}), \quad k = 0, 1, 2, \ldots,\end{gathered} \tag{2.6}$$

where $\overset{(k)}{\epsilon}_r$, $\overset{(k)}{s}_\theta$, etc., are the coefficients of the expansion equations $\epsilon_r = \sum_k z^k\overset{(k)}{\epsilon}_r$, $s_\theta = \sum_k z^k\overset{(k)}{s}_\theta$, etc., $(k = 0, 1, 2, \ldots)$ respectively.

According to the consideration in Chapt. 3 of [17], we make the following assumptions as a useful and reasonable guide to formulate the plate equations:

†The parameters λ and λ_h thus defined are more convenient for practical plate problems of variable thickness than those in [17].

$$\max_{S_0}(\epsilon_{2k}, \gamma_{2k-1}, \delta_{2k}) < \epsilon_0 \leqq \lambda^2 \leqq \lambda_B^2, \qquad \lambda_h < \lambda,$$
$$\max_{S_0}(s_{2k}, t_{2k-1}, u_{2k-2}) < s_0, \qquad k = 1, 2, 3, \dots, \tag{2.7†}$$

where λ_B is a universal constant which is chosen in such a way that the two-dimensional solution is valid with sufficient accuracy.‡

The material of the plate is assumed to be governed by the von Mises yield criterion and the constitutive relationship adopted by Hibbitt *et al.* [18]. Further we assume

$$H' \geqq s_0, \tag{2.8}$$

in which

$$\frac{1}{H'} = \frac{1}{E_T} - \frac{1}{E}, \tag{2.9}$$

where E is Young's modulus and E_T is the slope of the uniaxial stress-strain curve. The assumption (2.8) is widely applicable to many materials such as non-ferrous metals. The usual hardening materials stated mathematically by the Ramberg–Osgood equation are typical examples.

The axisymmetric equations based on (2.7) and (2.8) will be given below.

2.3 *Equation of equilibrium*

The state of stress in the absence of body and inertia forces will be considered. The mechanical boundary condition is specified on the surface $r = r^*$ and the major surfaces $z = \pm h_0/2$ are free from applied forces. We will state the simplified field equations for thin plates of finite thickness in which λ and λ_h are large enough, so that one needs to take into account the lowest order component of the transverse shear stress.

We denote the Lagrangian stress tensors by t_r, t_θ/r^2 and t_{rz}. Final results will only be given for the field equations in terms of the Lagrangian stress tensors. The derivation of these equations is referred to Chapt. 4 of [17].

From (4.4) and Table 1 of [17] we have the following expressions in both elastic and plastic domains:

$$\Delta t_r = \Delta \overset{(0)}{t}_r + z^2 \Delta \overset{(2)}{t}_r + O(\lambda^4 \Delta s_0, \lambda\lambda_h \Delta s_0),$$
$$\Delta t_\theta = \Delta \overset{(0)}{t}_\theta + z^2 \Delta \overset{(2)}{t}_\theta + O(\lambda^4 \Delta s_0, \lambda\lambda_h \Delta s_0), \tag{2.10}$$

and

$$\Delta t_{rz} = O(\lambda^3 \Delta s_0, \lambda_h \Delta s_0), \qquad (\Delta \overset{(2)}{t}_r, \Delta \overset{(2)}{t}_\theta) = \frac{1}{h_0^2} O(\lambda^2 \Delta s_0). \tag{2.11}$$

where "Δ" indicates the increment. The transverse normal stress is still smaller than those stress components.

We introduce a parameter η defined by

$$\eta = \frac{1}{12} \max_{S_0} \left(\frac{h_0}{L^*}\right)^2, \tag{2.12}$$

where L^* is the wave length of the solution obtained by the usual plane stress theory (Chapt. 4 of [17]). We assume that $\Delta \overset{(0)}{t}_r$, $\Delta \overset{(0)}{t}_\theta$, $\Delta\epsilon_r$ $\Delta\epsilon_\theta$ and Δu can be expanded as absolutely convergent series of η. For example

$$\Delta \overset{(0)}{t}_r = \Delta_0 \overset{(0)}{t}_r + \eta \Delta_2 \overset{(0)}{t}_r + \cdots. \tag{2.13}$$

†The case where $\lambda^2 < \epsilon_0$ can also be treated if λ^2 is replaced by ϵ_0 in the error terms of the simplified equations (2.10), (2.38) and (2.39). However in many stress concentration problems the applicability of the plate theory depends exclusively on the value of λ.

‡In the elastic stress concentration problems the two-dimensional assumption is found to be valid for plates with $\lambda_B \leqq O(1)$ [15].

The relation between the Lagrangian and Kirchhoff stress components are

$$ {}_0\overset{(0)}{t}_r = \sqrt{({}_0A_r)}\,{}_0s_r, \qquad {}_0\overset{(0)}{t}_\theta = \sqrt{({}_0A_\theta)}\,{}_0s_\theta, \tag{2.14} $$

where ${}_0A_r$, ${}_0A_\theta$, ${}_0s_r$ and ${}_0s_\theta$ are the coefficients of η^0 in the corresponding expansion equations of A_r, A_θ, s_r and s_θ, respectively.

The equilibrium equation governing $\Delta_0\overset{(0)}{t}_r$ and $\Delta_0\overset{(0)}{t}_\theta$ is the same as those of the usual plane stress theory. It can easily be transformed into the equation in terms of the Kirchhoff stress tensors:

$$ \sqrt{(A_r)}\frac{d\Delta s_r}{dr}+\left[\sqrt{(A_r)}\left(\frac{1}{h_0}\frac{dh_0}{dr}+\frac{1}{r}\right)+\frac{d^2u}{dr^2}\right]\Delta s_r-\sqrt{(A_\theta)}\frac{\Delta s_\theta}{r} $$

$$ +s_r\frac{d^2\Delta u}{dr^2}+\left[s_r\left(\frac{1}{h_0}\frac{dh_0}{dr}+\frac{1}{r}\right)+\frac{ds_r}{dr}\right]\frac{d\Delta u}{dr}-s_\theta\frac{\Delta u}{r^2}=0. \tag{2.15} $$

Here and in the following A_r, A_θ, u, s_r and s_θ are occasionally written instead of ${}_0A_r$, ${}_0A_\theta$, ${}_0u$, ${}_0s_r$ and ${}_0s_\theta$ for simplicity.

The stress resultants in our theory are defined by

$$ n_r=\int_{-h_0/2}^{h_0/2} t_r dz, \qquad n_\theta=\int_{-h_0/2}^{h_0/2} t_\theta dz. \tag{2.16} $$

Boundary condition at $r=r^*$ is derived from (4.38) of [17] as follows:

$$ \Delta_0\overset{(0)}{t}_r=\frac{1}{h_0}\Delta n_r^*, \tag{2.17} $$

where Δn_r^* is the increment of the prescribed stress resultant at $r=r^*$.

2.4 *Constitutive equations*

In this subsection the equations for the coefficients of η^0 will be derived. They do not contain the transversely variable stress and strain components. It is shown that the constitutive equations adopted by Hibbitt *et al.*[18] can be transformed into the following relations

$$ \Delta s_r=f_{rr}\Delta\epsilon_r+f_{r\theta}\Delta\epsilon_\theta, \qquad \Delta s_\theta=f_{\theta r}\Delta\epsilon_r+f_{\theta\theta}\Delta\epsilon_\theta, \tag{2.18} $$

where

$$ f_{rr}=\frac{2\mu}{A_r}\sqrt{(G_0)}\left(\frac{1}{A_r}+\frac{\nu}{1-2\nu}B_r\right)+s_r(\Gamma_r-1)-\frac{2\mu}{S_0\sqrt{(G_0)}}\left(s_r-\frac{s_m}{3A_r}\right)\left(s_r-\frac{1}{3}s_mB_r\right), $$

$$ f_{r\theta}=\frac{2\nu}{1-2\nu}\frac{\mu}{A_r}\sqrt{(G_0)}B_\theta+s_r(\Gamma_r+1)-\frac{2\mu}{S_0\sqrt{(G_0)}}\left(s_r-\frac{s_m}{3A_r}\right)\left(s_\theta-\frac{1}{3}s_mB_\theta\right), \tag{2.19} $$

$$ f_{\theta r}=\frac{2\nu}{1-2\nu}\frac{\mu}{A_\theta}\sqrt{(G_0)}B_r+s_\theta(\Gamma_\theta+1)-\frac{2\mu}{S_0\sqrt{(G_0)}}\left(s_\theta-\frac{s_m}{3A_\theta}\right)\left(s_r-\frac{1}{3}s_mB_r\right), $$

$$ f_{\theta\theta}=\frac{2\mu}{A_\theta}\sqrt{(G_0)}\left(\frac{1}{A_\theta}+\frac{\nu}{1-2\nu}B_\theta\right)+s_\theta(\Gamma_\theta-1)-\frac{2\mu}{S_0\sqrt{(G_0)}}\left(s_\theta-\frac{s_m}{3A_\theta}\right)\left(s_\theta-\frac{1}{3}s_mB_\theta\right), $$

with

$$ S_0=\frac{2}{3G_0}\left(1+\frac{H_0'}{3\mu}\right)(A_r^2s_r^2-A_rA_\theta s_rs_\theta+A_\theta^2s_\theta^2), \tag{2.20} $$

$$ s_m=A_rs_r+A_\theta s_\theta, \qquad G_0=(1+2r_z)A_rA_\theta, \tag{2.21} $$

$$B_r = \frac{1}{A_r} + \frac{\Gamma_r}{1+2r_z}, \qquad B_\theta = \frac{1}{A_\theta} + \frac{\Gamma_\theta}{1+2r_z}, \tag{2.22}$$

and where μ is the shear modulus, ν Poisson's ratio and

$$\Gamma_r = -2c\left[\frac{\nu}{A_r}\sqrt{(G_0)} + \frac{1-2\nu}{3S_0\sqrt{(G_0)}} s_m\left(s_r - \frac{s_m}{3A_r}\right)\right],$$

$$\Gamma_\theta = -2c\left[\frac{\nu}{A_\theta}\sqrt{(G_0)} + \frac{1-2\nu}{3S_0\sqrt{(G_0)}} s_m\left(s_\theta - \frac{s_m}{3A_\theta}\right)\right], \tag{2.23}$$

$$\frac{1}{c} = \frac{2}{1+2r_z}\left[(1-\nu)\sqrt{(G_0)} - \frac{1-2\nu}{9S_0\sqrt{(G_0)}} s_m^2\right]. \tag{2.24}$$

If we denote the equivalent stress by s_E which corresponds to the slope H_0' of the uniaxial stress-strain curve, we have

$$s_E^2 = \frac{3S_0}{2\left(1+\dfrac{H_e'}{3\mu}\right)}. \tag{2.25}$$

The increment of the transverse normal strain is expressed by

$$\Delta\epsilon_z = (1+r_z)\Delta r_z = \Gamma_r\Delta\epsilon_r + \Gamma_\theta\Delta\epsilon_\theta. \tag{2.26}$$

The differential equations in terms of the displacement increment $\Delta u (\equiv \Delta_0 u)$ is obtained by the substitution of (2·18) into (2·15) and with the aid of (2.3).

2.5 *Transversely variable stress and strain components*

It is convenient for the treatment of the transverse components to employ a new coordinate $\bar{z}$ which denotes the perpendicular distance from the deformed middle plane. We therefore introduce a new coordinate system $(\bar{r}, \theta, \bar{z})$ in addition to (r, θ, z) in which $(\bar{r}, \theta, 0) = (r, \theta, 0)$. Using the estimates of the higher order strain components given in Table 1 of [17], we have

$$\frac{\partial\bar{r}}{\partial r} = 1 + O(\lambda^2\epsilon_0), \qquad \frac{\partial\bar{r}}{\partial z} = O(\lambda\epsilon_0),$$

$$\frac{\partial\bar{z}}{\partial r} = z\frac{dr_z}{dr} + O(\lambda^3\epsilon_0, \lambda^2\lambda_h\epsilon_0), \qquad \frac{\partial\bar{z}}{\partial z} = 1 + r_z + O(\lambda^2\epsilon_0, \lambda\lambda_h\epsilon_0). \tag{2.27}$$

If we denote the Kirchhoff stress tensors at a generic point of the plate defined in the new and original coordinate systems by putting a bar and a hat respectively over the symbols, we find from (2.27) the following relations

$$\Delta\bar{s}_r = \Delta\hat{s}_r + O[\lambda^2\Delta(\epsilon_0 s_0)], \qquad \Delta\bar{s}_\theta = \Delta\hat{s}_r + O[\lambda^2\Delta(\epsilon_0 s_0)],$$

$$\Delta\bar{s}_{rz} = \Delta\hat{s}_{rz} + z\Delta\left(\frac{dr_z}{dr}s_r\right) + O[\lambda^3\Delta(\epsilon_0 s_0), \lambda_h\Delta(\epsilon_0 s_0)], \tag{2.28}$$

$$\Delta\bar{s}_z = \Delta\hat{s}_z + O[\lambda^2\Delta(\epsilon_0^2 s_0)].$$

The stress components $\Delta\bar{s}_r$, $\Delta\bar{s}_\theta$ and $\Delta\bar{s}_z$ can be approximated by $\Delta\hat{s}_r$, $\Delta\hat{s}_\theta$ and $\Delta\hat{s}_z$ respectively within the magnitudes of the error terms mentioned above.

By taking account of the error terms in (2.10) and by using (2.12), (2.13) and (2.14), it can be shown that

$$\Delta\hat{s}_r = \Delta_0 s_r + \left(z^2 - \frac{h_0^2}{12}\right)\Delta s_{2r}, \qquad \Delta\hat{s}_\theta = \Delta_0 s_r + \left(z^2 - \frac{h_0^2}{12}\right)\Delta s_{2\theta}, \tag{2.29}$$

where $h_0^2\Delta s_{2_r} = h_0^2\Delta \overset{(2)}{t}_r + O(\lambda^4\Delta s_0, \lambda\lambda_h\Delta s_0)$ and $h_0^2\Delta s_{2_\theta} = h_0^2\Delta \overset{(2)}{t}_\theta + O(\lambda^4\Delta s_0, \lambda\lambda_h\Delta s_0)$. These second order terms are also interpreted as the components proportional to η when the mechanical boundary conditions are specified on the rim of the hole (Chapt. 4 of [17]).

The strain components corresponding to (2.10) are as follows:

$$\Delta\epsilon_r + z^2\Delta\epsilon_{2r} + O(\lambda^4\Delta\epsilon_0, \lambda\lambda_h\Delta\epsilon_0), \Delta\epsilon_\theta + z^2\Delta\epsilon_{2\theta} + O(\lambda^4\Delta\epsilon_0, \lambda\lambda_h\Delta\epsilon_0),$$

where by (4.35) of [17]

$$\Delta\epsilon_{2r} = -\frac{1}{2}\frac{d^2\Delta r_z}{dr^2}, \qquad \Delta\epsilon_{2\theta} = -\frac{1}{2r}\frac{d\Delta r_z}{dr}. \tag{2.30}$$

The constitutive equations for the second order in-plane components are also found from the exact three-dimensional equations. They are

$$\begin{aligned} \Delta s_{2r} &= f_{rr}\Delta\epsilon_{2r} + f_{r\theta}\Delta\epsilon_{2\theta} + f_{2rr}\Delta\epsilon_r + f_{2r\theta}\Delta\epsilon_\theta, \\ \Delta s_{2\theta} &= f_{\theta r}\Delta\epsilon_{2r} + f_{\theta\theta}\Delta\epsilon_{2\theta} + f_{2\theta r}\Delta\epsilon_r + f_{2\theta\theta}\Delta\epsilon_\theta, \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} f_{2rr} &= \frac{2\nu}{1-2\nu}\mu\Gamma_{2r} - \frac{2\mu}{S_0}\left[\left(s_r - \frac{s_m}{3}\right)(R_{2r} - S_2R_r) + \left(s_{2r} - \frac{s_{2m}}{3}\right)R_r\right], \\ f_{2r\theta} &= \frac{2\nu}{1-2\nu}\mu\Gamma_{2\theta} - \frac{2\mu}{S_0}\left[\left(s_r - \frac{s_m}{3}\right)(R_{2\theta} - S_2R_\theta) + \left(s_{2r} - \frac{s_{2m}}{3}\right)R_\theta\right], \\ f_{2\theta r} &= \frac{2\nu}{1-2\nu}\mu\Gamma_{2r} - \frac{2\mu}{S_0}\left[\left(s_\theta - \frac{s_m}{3}\right)(R_{2r} - S_2R_r) + \left(s_{2\theta} - \frac{s_{2m}}{3}\right)R_r\right], \\ f_{2\theta\theta} &= \frac{2\nu}{1-2\nu}\mu\Gamma_{2\theta} - \frac{2\mu}{S_0}\left[\left(s_\theta - \frac{s_m}{3}\right)(R_{2\theta} - S_2R_\theta) + \left(s_{2\theta} - \frac{s_{2m}}{3}\right)R_\theta\right], \end{aligned} \tag{2.32}$$

with

$$S_2 = \frac{2}{3}\left(1 + \frac{H_0'}{3\mu}\right)[2(s_rs_{2r} + s_\theta s_{2\theta}) - s_rs_{2\theta} - s_\theta s_{2r}] + \frac{2H_h'}{9\mu}(s_r^2 - s_rs_\theta + s_\theta^2), \tag{2.33}$$

$$\begin{aligned} &R_r = s_r - \frac{1}{3}s_mb_r, \qquad R_\theta = s_\theta - \frac{1}{3}s_mb_\theta, \\ &R_{2r} = s_{2r} - \frac{1}{3}s_{2m}b_r, \qquad R_{2\theta} = s_{2\theta} - \frac{1}{3}s_{2m}b_\theta, \qquad s_{2m} = s_{2r} + s_{2\theta}, \end{aligned} \tag{2.34}$$

$$b_r = 1 + \Gamma_r, \qquad b_\theta = 1 + \Gamma_\theta, \tag{2.35}$$

and

$$\begin{aligned} \Gamma_{2r} &= -\frac{2(1-2\nu)}{9S_0}c[s_{2m}(2s_r - s_\theta) + s_m(2s_{2r} - s_{2\theta}) - 2s_ms_{2m}\Gamma_r], \\ \Gamma_{2\theta} &= -\frac{2(1-2\nu)}{9S_0}c[s_{2m}(2s_\theta - s_r) + s_m(2s_{2\theta} - s_{2r}) - 2s_ms_{2m}\Gamma_\theta]. \end{aligned} \tag{2.36}$$

The slope $H' = H_0' + h_0^2H_h'/4$ corresponds to the equivalent stress s_{2E} given by

$$s_{2E} = s_E + \frac{1}{4}h_0^2[2(s_rs_{2r} + s_\theta s_{2\theta}) - s_rs_{2\theta} - s_\theta s_{2r}]. \tag{2.37}$$

It is easily shown that the deformed thickness h is approximately connected with h_0 by $h = r_z h_0$. From (4.1) and (4.15) of [17] the maximum shear stress through the thickness is given by

$$\Delta \bar{s}_{rz} = \frac{1}{2}\frac{dh_0}{dr}\Delta s_r + \frac{h_0}{2}\Delta\left(\frac{dr_z}{dr}s_r\right) + O(\lambda^5 \Delta s_0, \lambda^2\lambda_h \Delta s_0). \tag{2.38}$$

The transverse normal stress is less important than the other stresses for thin plates since by (4.8) of [17]

$$\Delta \bar{s}_z = O[\lambda^2 \Delta(\epsilon_0 s_0), \lambda\lambda_h \Delta s_0]. \tag{2.39}$$

3. EXPANSION OF A HOLE IN AN INFINITE PLATE

3.1 *Statement of problem and method of solution*

We consider a thin infinite plate of the initial thickness h_0 with a cylindrical hole of the initial radius a. Let a gradually increasing pressure be applied over the surface of the hole.

For convenience in obtaining the solution, we introduce the following non-dimensional quantities:

$$\begin{gathered} r = a\rho, \quad u = aU, \quad s_r = \mu S_\rho, \quad s_\theta = \mu S_\theta, \quad H_0' = \mu H_0, \; s_E = \mu \Sigma_0, \\ h_0{}^2 s_{2r} = 4\mu S_{II\rho}, \quad h_0{}^2 s_{2\theta} = 4\mu S_{II\theta}, \quad \bar{s}_{rz} = \mu \bar{S}_{\rho z}. \end{gathered} \tag{3.1}$$

By taking account of (2.17), (2.29) and the relation between the Kirchhoff stress tensor and the physical component of the stress in the deformed state, the following boundary condition can be used at $\rho = 1$ for simplicity:

$$\Delta S_\rho = -\Delta Q(\text{say}). \tag{3.2}$$

A second order differential equation in terms of ΔU is derived from (2.15) through (2.18), (2.3) and (3.1). The boundary condition (3.2) is also expressed in terms of ΔU.

Now we shall suppose that the material of the plate shows the non-linear stress-strain behavior of the Ramberg–Osgood type:

$$\epsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{B}\right)^N, \tag{3.3}$$

where ϵ and σ are the logarithmic strain and the true stress respectively, B is a material constant and N is a function of strain hardening. In this paper we will take an alluminium alloy [20] ($B/E = 7{\cdot}23 \times 10^{-3}$, $N = 10$) as an example of the materials having the non-linear stress-strain relation. Therefore H_0 is numerically equal to

$$H_0 = \frac{\Sigma_0}{10}\left(\frac{1}{\omega \Sigma_0}\right)^{10} \quad \text{where} \quad \omega = 51{\cdot}8. \tag{3.4}$$

Poisson's ratio has been taken equal to 1/3. Fig. 1 shows the uniaxial stress-strain curve for this material.

Following [10–13] the initial thickness of the plate is assumed to be given by $h_0 = \alpha r^n$ where α and n are constants. Here n is assumed to satisfy $n \leqq 0$.

Since it is difficult to solve the non-linear differential equations explicitly, we shall apply the finite difference method. A uniform mesh of points is placed in the interval $1 \leqq \rho \leqq k$:

$$\rho_i = id + 1, \qquad i = 0, 1, 2, \ldots, l, \qquad d = \frac{k-1}{l}, \tag{3.5}$$

where $\rho \geqq k$ is the domain in which the deformation is so small that the linear elasticity can be used with sufficient accuracy. The displacement $U(\rho)$ is then approximated by a mesh function U_i. We employ here a difference quotient to approximate a differential quotient with an error $O(d^2)$ as $d \to 0$. The finite difference equivalents of the differential equation and the boundary conditions can be written in the form

$$f_q(U_i, \Delta U_j, \Delta Q) = 0, \qquad i, j, q = 0, 1, 2, \ldots, l, \tag{3.6}$$

where $f_0 = 0$ corresponds to the boundary condition (3.2). As for the condition $f_1 = 0$ we use the following relation obtained from the linear elasticity

$$(1 + \nu\chi)\Delta S_\rho - (\nu + \chi)\Delta S_\theta = 0, \tag{3.7}$$

where

$$\chi = \frac{1}{2}[-n - \sqrt{(n^2 + 4(1 - \nu n))}]. \tag{3.8}$$

The non-linear algebraic equations (3.6) are solved iteratively starting from some initial estimates of the mesh functions for a given value ΔQ.

The stress increments $\Delta S_{II\rho}$ and $\Delta S_{II\theta}$ are obtained from (2.32) after the displacement ΔU is determined. The transverse shear stress is also found from (2.38) and (3.1) as follows:

$$\bar{S}_{\rho z} = \frac{1}{2}\left(\frac{h_0}{a}\right)\left(\frac{dr_z}{d\rho} + \frac{n}{\rho}\right)S_\rho. \tag{3.9}$$

3.2 *Results and discussions*

The numerical computation is executed by the computer NEAC 2200–700 for the plate having the ratio $h_0/a = 0{\cdot}5$.† The results for $k = 5{\cdot}5$ and $l = 99$ are shown below. From a series of test calculations it was observed that the domain $k \geqq 5{\cdot}5$ is purely elastic for ranges of the loads considered. For comparison the plane stress solution of the classical elastic-plastic theory based on the geometrically linear assumption was also calculated.

Figures 2–4 show the solutions for the plate of uniform thickness ($n = 0$). When the pressure Q is small the equation (3.7) is satisfied with sufficient accuracy except in the neighborhood of the rim of the hole as shown in Fig. 2a. The transversely variable stress components are also graphed in Figs. 2b and 2c. These components are not large compared with the first order in-plane stresses for the plate of the uniform thickness. However they become larger together with the value of the ratio h_0/a.‡ Representative graph of the ratio $(h - h_0)/h_0$ is presented as the function of ρ in Fig. 4.

In Figs. 5–7 the solutions for the variable thickness plate ($n = -0{\cdot}5$) are shown. From Figs. 5a and 5b the transverse shear stress is found to be large which has been ignored in the usual plane stress theory. That stress has to be taken into account for plates in which the ratio h_0/a is about 0·5 or more.

It is also found from these figures that the classical solutions are valid only for $Q < 10^{-2}$ except for the transversely variable components.

Applicability of the two-dimensional equations (one-dimensional equations for axisymmetrical problems) depends on the parameter λ_B. In many two-dimensional problems of the finite thickness plate λ and/or λ_h plays an important part. Conversely for the problems in which the stress concentrates rapidly in the radial direction, sufficiently small values are therefore required of the ratio h_0/a. It should be noted that the radial change of the thickness due to the deformation is characterized by the parameter λ.

†For the theory of finite thickness plates the ratio h_0/a has to be specified.

‡It is not easy to evaluate the numerical value of the bound λ_B. If we can apply the results in [15] to elastic-plastic problems, λ_B is of $O(1)$ so that the two-dimensional theory cannot be valid for plates with $h_0/a \gg 1$. The parameter λ is approximately given by

$$\lambda = \frac{h_0 \max_{S_0} |S_{\rho,\rho}, S_{\theta,\rho}|}{a \max_{S_0} |S_\rho, S_\theta|}.$$

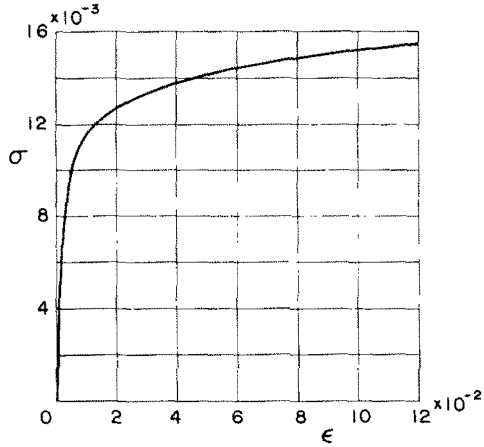

Fig. 1. Uniaxial stress-strain curve.

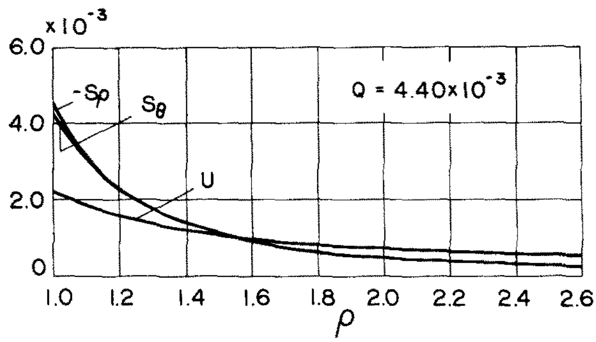

Fig. 2a. Stress and displacement distributions for $n = 0$.

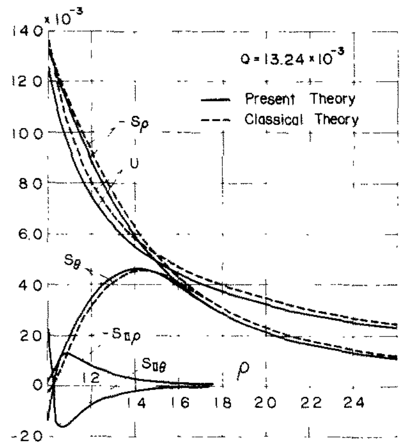

Fig. 2b. Stress and displacement distributions for $n = 0$.

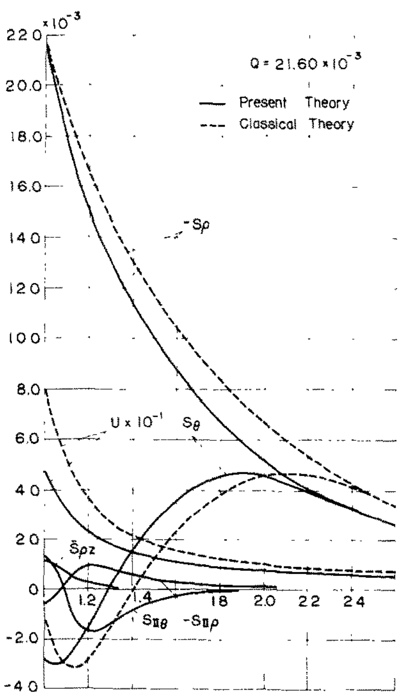

Fig. 2c. Stress and displacement distributions for $n = 0$.

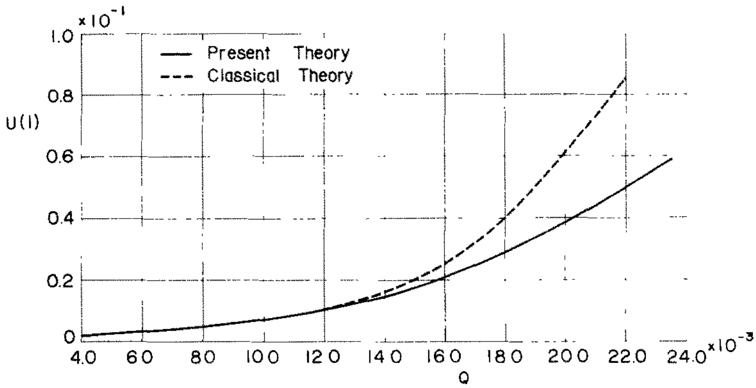

Fig. 3. Radial displacement at the edge of the hole for $n = 0$.

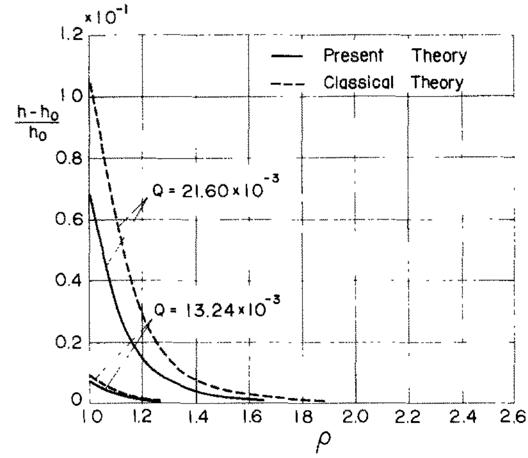

Fig. 4. Variation of thickness for $n = 0$.

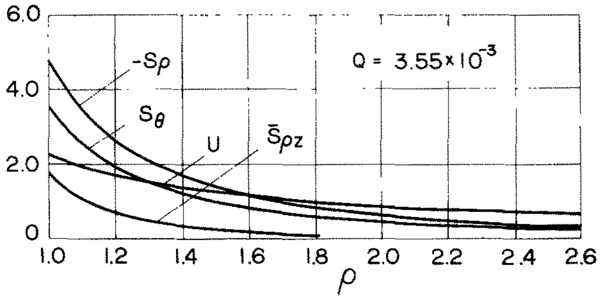

Fig. 5a. Stress and displacement distributions for $n = -0{\cdot}5$.

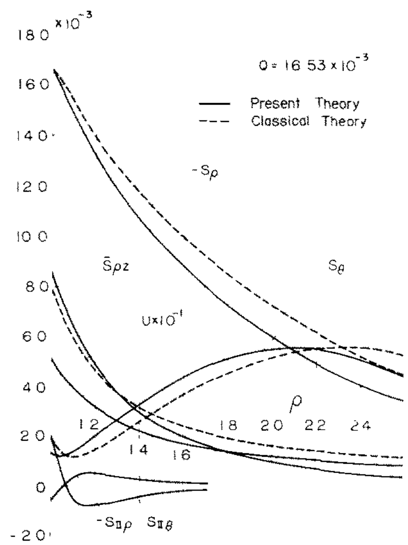

Fig. 5b. Stress and displacement distributions for $n = -0{\cdot}5$.

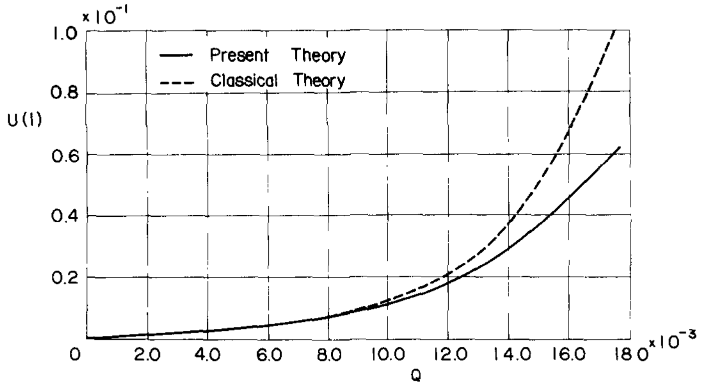

Fig. 6. Radial displacement at the edge of the hole for $n = -0{\cdot}5$.

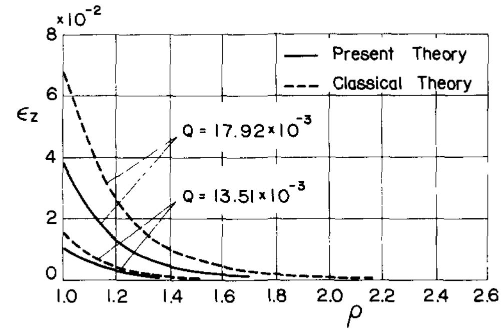

Fig. 7. Variation of transverse normal strain for $n = -0{\cdot}5$.

Finally it is noted that the solutions just obtained may involve some error in the close neighborhood of the boundary. This is inherent in the two-dimensional theory. However, if the distribution of the applied pressure over the thickness expressed in terms of the Kirchhoff stress is considered to be the same as that of the normal stress $\hat{s}_r$ on the boundary, the solutions are valid at any point of the plate with $\lambda \leqq \lambda_B$.

Acknowledgements—The author wishes to express his gratitude to Professor O. Tamate for his valuable advice and useful discussions in this investigation. Appreciation is also extended to Mr. T. Shōji for his assistance.

REFERENCES

1. G. I. Taylor, The formation and enlargement of a circular hole in a thin plastic sheet. *Q. J. Mech. Appl. Math.* **1**, 103 (1948).
2. R. Hill, *The Mathematical Theory of Plasticity*, p. 307. Oxford University Press (1950).
3. W. Prager, On the use of singular yield conditions and associated flow rules. *J. Appl. Mech.* **20**, 317 (1953).
4. P. G. Hodge, Jr. and R. Sankaranarayanan, On the finite expansion of a hole in a thin infinite plate. *Q. Appl. Math.* **16**, 73 (1958).
5. R. P. Nordgren and P. M. Naghdi, Finite twisting and expansion of a rigid/plastic plate. *J. Appl. Mech.* **30**, 605 (1963).
6. R. P. Nordgren and P. M. Naghdi, Loading and unloading solutions for an elastic/plastic annular plate in the state of plane stress under combined pressure and couple. *Int. J. Engng Sci* **1**, 33 (1963).
7. J. M. Alexander and H. Ford, On expanding a hole from zero radius in a thin infinite plate. *Proc. R. Soc.* **A226**, 543 (1954).
8. T. G. Rogers, The finite expansion and subsequent unloading of a hole in an elastic-plate of initially varying thickness. *Q. J. Mech. Appl. Math.* **20**, 137 (1967).
9. S. Nemat-Nasser, Finite expansion of a hole in a rigid-workhardening disk of initially non-uniform initial thickness. *J. Mech. Phys. Solids.* **16**, 196 (1968).
10. J. M. Chern, Expansion of a hole in a disk with kinematic hardening. *Int. J. Mech. Sci.* **11**, 623 (1969).
11. J. M. Chern and S. Nemat-Nasser, The enlargement of a hole in a rigid-workhardening disk of non-uniform initial thickness. *J. Mech. Phys. Solids* **17**, 271 (1969).
12. J. M. Chern and S. Nemat-Nasser, Similarity solutions for expansion of a hole in anisotropic plastic disks. *ZAMP* **21**, 66 (1970).
13. J. M. Chern and S. Nemat-Nasser, Twisting and expansion of a hole in a non-uniform disk. *Int. J. Solids Structures* **6**, 829 (1970).
14. E. H. Mansfield, Optimum variable thickness reinforcement around a circular hole in a flat sheet under radial tension. *Q. J. Mech. Appl. Math.* **24**, 499 (1971).

15. H. Abé, On the transverse shear stress in the finite deformation theory of thin elastic plates (Part I Basic theory). *Technology Reports Tohoku Univ.* **38**, 503 (1973).
16. H. Abé, On the transverse shear stress in the finite deformation theory of thin elastic plates (Part II Applications). *Technology Reports Tohoku Univ.* **38**, 519 (1973).
17. H. Abé, Finite deformation theory for the stretching of a thin elastic-plastic plate of finite thickness. *Technology Reports Tohoku Univ.* **38**, 539 (1973).
18. H. D. Hibbitt, P. V. Marcal and J. R. Rice, A finite element formulation for problems of large strain and large displacement. *Int. J. Solids Structures* **6**, 1069 (1970).
19. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., p. 206. Cambridge University Press (1934).
20. F. R. Shanley, *Mechanics of Materials*, p. 20. McGraw-Hill (1967).